Uncontested Incumbents and Incumbent Upsets*

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Abstract

This paper presents a model of elections that explores the relationship between incumbent upsets and incumbents running unopposed. A valence-disadvantaged candidate (a potential challenger), who possesses private information about the extent of this disadvantage, first decides whether to challenge a valence-advantaged candidate (the incumbent). If the election is contested, the two candidates engage in policy competition. In equilibrium, the incumbent will never be certain of the strength of a challenger she faces, and she may risk losing by proposing a less moderate policy for the possibility of winning at a policy more to her liking. The model demonstrates that analysts risk mischaracterizing candidate behavior – even in races in which an incumbent defeats a challenger – by ignoring the possibility that an advantaged candidate may lose due to "rational complacency." Comparative statics may switch signs if analyses exclude uncontested elections, and reducing the frequency of uncontested elections entails the trade-off for voter welfare of more extreme platforms in contested elections.

Keywords: Uncontested elections; upsets; incumbency advantage; valence; contests

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Introduction

In the 2014 midterm elections, Steve Southerland, a two-term Republican incumbent representing Florida's 2$^{\text{nd}}$ Congressional District, faced a challenge from political upstart Gwen Graham. At the outset, the Republican establishment and Southerland himself considered the seat to be safe. Yet Graham ran an energetic and decidedly ideologically moderate campaign, while Southerland did little to distance himself from Republican obstinance during the Obama presidency. When asked about his campaign's slow start and the early lead the centrist Graham had taken in the polls, Southerland explained away his lack of concern by saying, "It's easy to score touchdowns when the defense isn't on the field," reportedly leaning back against his pick-up truck as if to underscore the recognition of his own complacency. Soon after, Southerland and his party would begin pouring resources into the race. It would be too late to make up for early mistakes, though, and Graham would go on to defeat Southerland that November (Sherman 2014).

Southerland should have been well-acquainted with the pitfalls of incumbent complacency, especially around the failure to moderate ideologically. Just four years earlier, then-political neophyte Southerland defeated seven-term incumbent Allen Boyd in the 2010 midterm election in the same district. Boyd, a Blue-Dog Democrat, entered the race with a four-to-one fundraising advantage, but in addition to an unclear stance on the Affordable Care Act, Boyd abandoned past fiscal conservatism and embraced President Obama's stimulus spending. Reflecting his uncharacteristic tack leftwards, signs reading "Blue Dog = Lap Dog" emerged around the district, impugning his credentials as a moderate (Ward 2010).

In neither of these instances of incumbent upsets does it seem right to infer that the current officeholders did not benefit from the well-documented incumbency advantage. The fact that Southerland and his party considered the seat to be safe suggests the candidate had no *ex ante* liabilities, and Boyd was the picture of an advantaged incumbent, with over a decade in office and a massive warchest. Further, each might have suspected his challenger was likely to be strong from the opponent's decision to contest the race in the first place. Instead, it appears each made a calculated decision to run closer to their own ideological preferences than those of the district. While they may appear complacent in retrospect, to treat the decisions as irrational does a disservice both to analyses of these particular elections and, more broadly, to analyses of elections with an incumbent or a candidate otherwise thought to possess an advantage.

The model presented in this paper offers an explanation for why an incumbent who could ensure victory for herself with a campaign targeted at the median voter would rationally risk defeat and adopt a more ideologically extreme stance. Further, it seeks to do so without assuming the decision of a potential challenger to contest the election is "as if random" and without consequence for candidate behavior in the election. In so doing, it offers one account of the way in which uncontested elections, incumbents retaining their seats, and challengers upsetting incumbents may all arise in the same equilibrium. The unified explanation of these three phenomena provides a deeper understanding of the relationship among these outcomes, and it leads to a range of new insights regarding the evaluation of electoral competition and voter welfare.

A potential challenger, who knows if he is a high- or low-valence type, first decides whether to enter the election or not. The incumbent, while possessing at least a weak valence advantage over either strength challenger, is uncertain as to the type of the potential challenger. If the challenger enters, the two, policy-motivated candidates engage in policy competition. The resulting game is essentially a first-price auction with a privately-informed bidder, an entry decision, and – because losing candidates care about the winning policy – positive spillovers.

In equilibrium, the two types of challenger never fully separate with respect to entry. The intuition for this result centers on the interaction of the uncertainty about challenger strength with the challenger's strategic entry decision. If an incumbent believes she will only face strong challengers, a weak challenger will induce significantly more moderation in policy from an incumbent simply by entering than the policy moderation such a challenger might expect based strictly on her viability as a candidate. If, however, an incumbent is quite sure she will only be challenged by weak candidates, a strong candidate becomes a wolf in sheep's clothing, able to capitalize on an incumbent's complacency and potentially achieve an upset victory. Rather than signaling, the challenger achieves obfuscation through entry. Specifically, while the strong type always enters, the weak type enters with some positive probability. Put another way, the strong hand never folds, and the weak hand sometimes bluffs.

Uncontested incumbents, upset incumbents, and retained incumbents all arise in the equilibrium case of greatest interest. Importantly, the upsets result not from exceptions to the rule of incumbency advantage, but rather because the incumbent's uncertainty persists, and she may wish to push the limits of her advantage. Indeed, in this equilibrium case, a greater likelihood of contested elections implies a higher likelihood that the incumbent faces a weak challenger. In turn, this leads the incumbent to display higher levels of rational complacency when challenged. This translates into more incumbent upsets among contested elections but also into less moderate policy proposals in these same elections, effects that correspond to increases and decreases in voter welfare, respectively. Several exogenous changes that increase the frequency of contested elections thus also entail a trade-off of decreased moderation within contested elections.

It is worth noting at the outset two features of the model that facilitate broader interpretations. First, the valence-advantaged candidate need not be an incumbent. The interpretation of the valence-advantaged candidate as an incumbent is natural, and incumbency is an important phenomenon in the study of U.S. politics that the model may speak to. Yet the intuition about an advantaged candidate exhibiting "rational complacency" may apply equally well to candidates deemed frontrunners at the outset of a campaign due to political legacy, previous experience holding other public office, or party incumbency. Second, many of the model's dynamics would carry through if the assumptions of spatial competition and policy-motivated candidates were replaced with costly competition over expenditures/effort and candidates who prefer, even when they lose, that their opponents incur greater cost. Additional insights arise when grounding the model in policy competition, yet the core mechanisms are not reliant on the use of binding policy proposals. To fix ideas, the paper proceeds by referring to candidates as an incumbent and a (potential) challenger, as well as by employing policies as the substrate of competition.

The next section discusses the incorporation of incumbent uncertainty into the model, noting how this approach differs from and contributes to various related literatures. The following section details the model and derives the equilibrium cases. The remainder of the paper distills a number of insights from the model, including often counterintuitive implications for the evaluation of electoral competition and voter welfare. Specifically, results follow on the equilibrium dependence between the probability that incumbents run uncontested and the probability that challengers upset incumbents, the way in which individual strong challengers suffer even as it becomes more likely that a strong challenger will defeat an incumbent, and the tension for voter welfare between reducing uncontested elections and encouraging more moderate policy proposals in contested elections. A brief conclusion follows.

Incumbent Uncertainty About Challenger Strength

Two questions about the decision to challenge an incumbent have consistently motivated studies of upset victories, uncontested elections, and their significance for voter welfare. First, to what extent and to what effect do strong challengers wait for an open seat to run for office, the so-called "scare-off" effect of incumbency (Abramowitz 1991, Cox & Katz 1996, Jacobson 1989, Krasno & Green 1988, Stone, Maisel & Maestas 2004)? Second, why would weak candidates ever challenge a sitting office-holder (Banks & Kiewiet 1989, Canon 1993)? The reasoning implicit in these questions has a degree of intuitive appeal, and goes something as follows: given the well-documented incumbency advantage, strong challengers ought to increase their comparatively better chances at winning by competing on a more level playing field, i.e., entering an open-seat race;[1] meanwhile, weak challengers only stand to decrease their already low chances of winning by taking on a sitting official. This line of reasoning tacitly makes the strong assumption that an incumbent knows the strength of a challenger she faces.

The model presented in this paper suggests the logic underlying these common arguments about strategic entry may be incomplete in the presence of incumbent uncertainty about challenger strength. It demonstrates

[1] Hall & Snyder (2015) provide a thorough summary of the development of this received wisdom.

that if candidates care about the winning policy even if they lose, and if incumbents must adopt different policies to retain their seats when challenged than if running uncontested, then it need not be the case, as the studies above presume, that only strong or weak challengers enter. Initial uncertainty may persist, and the incumbent will never know with certainty whether she faces a strong challenger or a weak challenger in equilibrium. This remains true even if the challenger knows his type and has the opportunity to signal it through entry.

The initial informational advantage a challenger may possess about his strength could arise from a number of sources. The challenger need not be untested. A potential challenger will have better knowledge of his motivations for running (genuinely seeking office or just running to keep an incumbent honest), means (personal and political capital), and party support (financial and otherwise, e.g., promised appearances from high-profile party members on the campaign trail).

The incorporation of the potential challenger's entry decision plays a crucial role in all of the results, and along with the uncertainty differentiates this model from previous models with entry or valence-advantaged candidates. Earlier models of challenger entry (Feddersen, Sened & Wright 1990, Osborne 1993, Osborne 2000) did not speak to incumbency (or advantaged candidates, more generally). Models of simultaneous spatial competition in which candidates possess differing levels of valence as well as some element of uncertainty (Groseclose 2001, Aragones & Palfrey 2002, Aragones & Palfrey 2005, Hummel 2010) do not engage with strategic entry or asymmetric information about quality.

In particular, the confluence of asymmetric information and challenger entry sets up the potential for incumbent learning about challenger strength. A spate of recent, mostly theoretical scholarship examines some element of the strategic interaction and learning that occurs between candidates and voters (Ashworth 2005, Ashworth & Bueno de Mesquita 2006, Ashworth & Bueno de Mesquita 2008, Banks & Sundaram 1998, Bernhardt, Camara & Squintani 2011, Dewan & Hortala-Vallve 2017, Epstein & Zemsky 1995, Gordon, Huber & Landa 2007, Gordon & Landa 2009, Gordon, Huber & Landa 2009). Few, however, shed light directly on the potential signaling that may occur through challenger entry, which determines the value an incumbent may extract from her advantage[2] though (Carson 2005) demonstrates the pitfalls of performing empirical analyses that ignore the strategic interaction and learning that occurs between challengers and incumbents.

Carter & Patty (2015) study dynamic policy competition where candidates possess different valences and make an effort decision akin to entry. While their model shares several features with the model in this paper, the authors assume candidates to be vote-maximizing, rather than some combination of office- and/or

[2]It is noteworthy that Gordon and Landa as well as Epstein and Zemsky uncover a tendency for incumbents of different types to pool in the equilibria of their models, as is found for challengers of different strengths in this paper.

policy-motivated, a significant point of difference (see Patty (2002)). In the context of a model with entry, this distinction is crucial.

Rather, the game here is essentially a contest with spillovers, uncertainty, and entry. The candidates compete over offers of policy and valence to the median voter, the inputs into the median voter's score function. Accounting for her valence, a candidate wishes to offer a moderate enough policy to win. Conditional on winning, she prefers policies closer to her ideal point (i.e., farther from the median voter's ideal point). Conditional on losing, a candidate benefits from any moderation in her opponent's proposal. Although the equilibrium is reminiscent of those in the contests of Hirsch & Shotts (2015) and Baye, Kovenock & de Vries (2012), the asymmetry, entry, and especially the uncertainty present lead the development of equilibrium to more closely resemble those of the first-price auctions with private information studied in Maskin & Riley (2000).

The Model

The set of players consists of an incumbent, L, a (potential) challenger R, and a continuum of voters whose ideal points are strictly ordered in $\mathbb{R}$. Let $c \in \{L, R\}$ denote an arbitrary candidate, v denote an arbitrary voter, and V denote the median voter. The potential challenger ("he") is either a strong type (S) or a weak type (W). The Incumbent ("she") takes only one type, which may be thought of as strong.

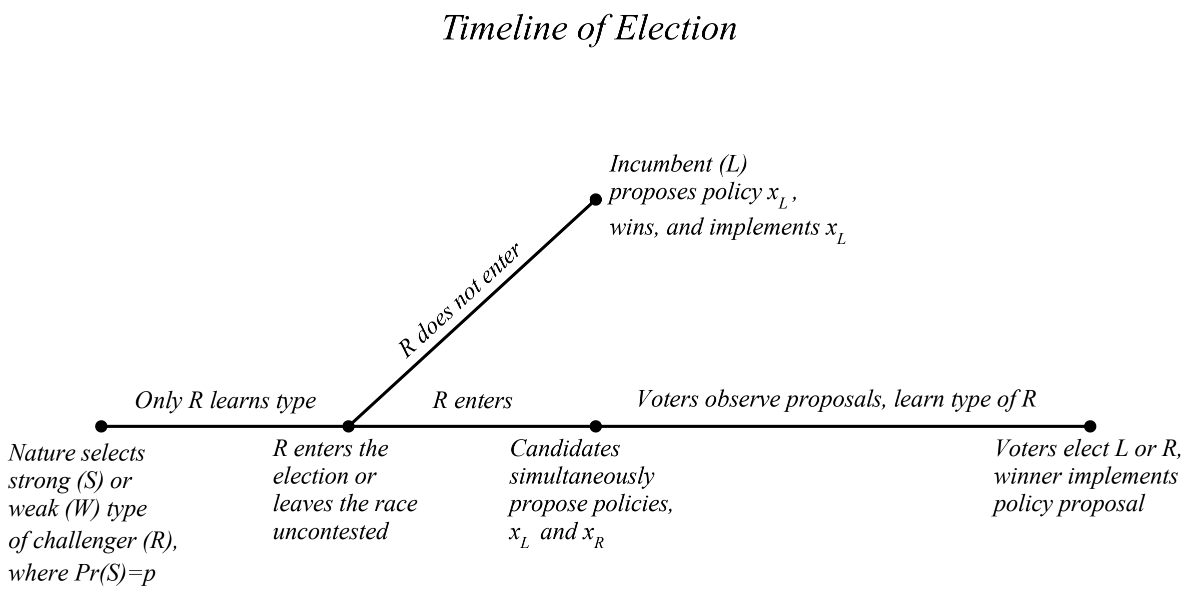
Nature first selects a type of challenger, $t \in \{S, W\}$, choosing a strong type ($t = S$) with probability $p \in (0, 1)$, where p reflects the strength of the pool of potential candidates. The challenger learns his type and then decides to enter the race or leave the seat uncontested. The entry decision of the challenger is immediately observable to all players, but the incumbent does not know the challenger's type. The incumbent and the challenger (if contesting the election) simultaneously propose a policy, $x_c \in \mathbb{R}$. Finally, all voters observe the policy proposals, learn the challenger's type, and cast a vote for exactly one of the candidates that entered the election. The candidate that garners a majority of votes wins the election and implements his/her platform.

Each player's utility from an implemented policy is strictly decreasing in the distance of the policy from the player's ideal point, $M_v, \forall v$, and $M_c, c = L, R$. Specifically, the disutility individual i receives from a winning policy x is given by $-|M_i - x|$. We assume that $M_L < M_V < M_R$. Both types of challenger share the same ideal point. For convenience, we set $M_V = 0$, hence $M_L < 0 < M_R$.[3]

Candidates incur a strictly positive cost upon entering the election, $C > 0$. Many interpretations are compatible with this quantity, including the signatures required to appear on a ballot, prices in the relevant

[3] In an extension below, the candidates are office-motivated as well as policy-motivated.

Figure 1: Order of Play



media market, and the geographic span or population of the constituency, all of which increase the cost of running.[4] The cost of running in the race enters additively into the candidates' utility. Payoffs to the candidates are then as follows:

$$U_L = \begin{cases} -|M_L - x_L| - C & \textit{if } L \textit{ wins} \\ -|M_L - x_R| - C & \textit{if } R \textit{ wins} \end{cases} \tag{1}$$

$$U_R = \begin{cases} -|M_R - x_R| - C & \textit{if } R \textit{ wins} \\ -|M_R - x_L| - C & \textit{if } L \textit{ wins and } R \textit{ enters} \\ -|M_R - x_L| & \textit{if } L \textit{ wins and } R \textit{ does not enter} \end{cases} \tag{2}$$

Voters derive non-policy utility specific to each (type of) candidate, in addition to disutility from the

[4]All (types of) candidates face the same cost of entry. Including some costs to entry are necessary if we want to see strategic entry (and strategic "non-entry"), as discussed in Callander (2008). Any separation that resulted from assuming greater costs on weaker challengers would not be strategic but rather an artifact of the different costs each type of R faced. The incumbent is assumed to have announced her intention to seek reelection, i.e., she does not have the choice not to run and forgo incurring C.

implementation of winning policies not at their ideal point. Some degree of valence is associated with each (type of) candidate and accrues to the voter if that candidate is elected, entering additively into each voter's utility function. Denote the incumbent's valence by A_L, the strong type of challenger's valence by $A_{R|S}$, and the weak type of challenger's valence by $A_{R|W}$. A voter's payoffs are specified below:

$$U_v = \begin{cases} -|M_v - x_L| + A_L & \textit{if } L \textit{ wins} \\ -|M_v - x_R| + A_{R|t} & \textit{if } R \textit{ wins and is of type } t \in \{S, W\} \end{cases} \tag{3}$$

Granting voters full information when deciding whom to vote for captures the idea that the challenger's type is revealed over the course of the campaign but after the candidates stake policy positions.

Imposing the ordering $A_L \geq A_{R|S} > A_{R|W}$ embodies the assumption that the incumbent possesses an advantage over either type of challenger, but a strictly larger advantage over the weak type, W, than over the strong type, S. Defining $\underline{a} := A_L - A_{R|S}$ and $\overline{a} := A_L - A_{R|W}$, the incumbent then possesses a valence advantage of $\underline{a}$ over the strong type of challenger and a valence advantage of $\overline{a}$ over the weak type of challenger, such that $\underline{a} < \overline{a}$. Let $a \in \{\underline{a}, \overline{a}\}$ denote an arbitrary such valence advantage. Given the conception of challenger strength as equivalent to the extent of the incumbent's valence advantage, nature is effectively choosing the size of L's net valence advantage, a, over R in the game's first move. To streamline the notation for the sake of exposition, in text the incumbent's valence is set equal to a strong challenger's valence ($A_L = A_{R|S} =: A$), and the weak challenger's valence is set equal to zero ($A_{R|W} = 0$).[5] Under this assumption, $\underline{a} = 0$ and $\overline{a} = A$.

Throughout, it is assumed the incumbent cannot be assured to win – against either type of challenger – by simply proposing her ideal point, M_L. This supposes that incumbents must propose a different platform to win the election if challenged than they would if they ran uncontested. It requires that the incumbent's ideal point not be too moderate (close to the median voter's) and/or that the incumbent's valence advantage not be too large, specifically, $M_L < -A$.

Preliminary Results

The solution concept for this sequential-move game with uncertainty is weak perfect Bayesian equilibrium. Only equilibrium strategies that do not include weakly dominated actions receive consideration. This implies

[5]The appendix derives equilibrium in the more general case, where $A_L \geq A_{R|S}$ and $A_{R|W} \geq 0$. To assume $A_L = A_{R|S}$ does make a degree of sense, though, as only strong challengers ascend to office in the model, so tomorrow's valence-advantaged incumbent may be today's high-valence potential challenger. The assumption that the weak challenger's valence is zero does not sacrifice any insight.

that voters will vote sincerely, which permits restricting attention to the median voter's decision (see Lemma 1 in Appendix A).[6] Henceforth, "the voter" will refer to the median voter, V.

Given the assumed ordering of ideal points, it is also weakly dominated for candidates to propose any policies that do not lie between their own ideal point and the ideal point of the median voter (see Lemma 2 in Appendix A). Neither type of R will take a position to the left of the median voter's ideal point ($x_R < M_V = 0$) or to the right of his own ideal point ($x_R > M_R$), nor will L propose a policy to the left of her ideal point ($x_L < M_L$) or to the right of the median voter's ideal point ($x_L > M_V = 0$). As such, it will always be the case that $x_L \in [M_L, 0]$ and $x_R \in [0, M_R]$.

The voter's decision rule if R does challenge L is straightforward, except in cases of indifference. Recall that the incumbent possesses a valence advantage of a over her challenger, and that while the valence advantage is known to R but not to L when each proposes a policy, it is revealed over the course of the election, so V will take the size of L's valence advantage over R into account when casting a vote. Given the payoffs specified above, V votes for L if

$$x_L + a > -x_R \tag{4}$$

and for R if the inequality were reversed.

The only cases of voter indifference that will occur with positive probability in equilibrium in a contested election involve the challenger converging entirely to the median voter's ideal point, 0. If R of type t proposes $x_R = 0$ and L proposes $x_L = A_{R|t} - A_L$ such that V is indifferent, equilibrium existence requires that V vote for L (see Lemma 3 in Appendix A). Such tie-breaking rules arise endogenously, in the sense of "endogenous sharing rules" (Simon & Zame 1990).

Remark Recalling the assumption that voters must vote for exactly one of the candidates that entered the election, if the incumbent is not challenged, she will propose her ideal point, M_L, and receive V's vote.

As such, if a potential challenger leaves the race uncontested, he receives the policy disutility associated with the incumbent's ideal point, M_L, as the winning platform, but he does not incur the cost of entry, C. Accordingly, let the "opportunity cost of contesting the election" be given by $C + M_L$. This quantity increases as the incumbent's ideal point becomes more moderate (recall $M_L < -A < M_V = 0$), which makes remaining out of the race more attractive, and as the cost of contesting the election increases, which makes contesting the election less attractive.

[6]The addition of valence to the model does nothing to affect the standard result that, in uni-dimensional policy settings, the support of the median voter is necessary and sufficient to achieve the support of a majority of voters.

If the incumbent knew which type she faced, she would only moderate her policy proposal as much as necessary to win, and she would have to converge farther when facing a strong type of challenger than when facing a weak type. The incumbent lacks this knowledge, however, unless the two types of potential challenger fully separate with respect to entry. Proposition 1 states that, in fact, the incumbent will never know exactly which type of challenger she faces. The two types at least partially pool on the entry decision, either both entering with strictly positive probability or neither entering at all. The central tension in the model, then, is that the incumbent is uncertain how much she must moderate in policy towards the median to win the election. If the incumbent guesses incorrectly which type she faces, she runs the risk of moderating more than she needed to in order to win against a weak challenger (incurring disutility from a policy farther from her ideal point) or losing the election to a strong type by not converging enough. In and of itself, the impossibility that the two types of challenger ever fully separate with respect to entry is an important, albeit straightforward, implication of the model.

Proposition 1. *The incumbent is never certain of the strength of a challenger who contests the election.*

If one type of candidate would be willing to enter, both would, so separation cannot occur on entry – no single-crossing condition obtains for the two candidates. Underlying this result is the benefit to losing challengers from being overestimated if the incumbent adopts a more moderate policy than she needs to in order to win, as well as the possibility of upsetting an incumbent who underestimates the strength of the challenger she faces and fails to propose a moderate enough policy to win. The proof, found in Appendix B, follows this same line of argument.

Characterizing Equilibrium Strategies and Cases

Lemmas 4 – 5 (see Appendix C), along with the results thus far, establish that three equilibrium cases may obtain, and all three cases do obtain for some open set in the parameter space. Figure 2 labels the region of the parameter space corresponding to each case. Specifically, either both types of potential challenger will enter with some strictly positive probability (cases 1-2) or neither type will enter (case 3). In cases where the potential challenger contests the election with positive probability, strong challengers will always enter, and weak challengers will either strictly randomize between entering and leaving the race uncontested (case 1) or enter with certainty (case 2).

Proposition 2 highlights how the equilibrium probabilities of a contested election and an upset victory vary in response to changes in the opportunity cost of contesting the election ($C + M_L$) and the probability the potential challenger is a strong type (p). For the latter outcome, the proposition highlights how the probability that an upset occurs responds differently to an increase in C or M_L depending on whether it is measured unconditionally and conditionally on the election being contested. The proposition follows immediately from Proposition 2', which may be found in Appendix D and which specifies the exact parameter values, the distributions of policy proposals over which the candidates randomize, and the voter strategies that characterize the equilibrium cases. Lemma 6 demonstrates that, when the candidates randomize, they do so according to distributional strategies (in the sense of Milgrom & Weber (1985)).

The distributions of proposals take a somewhat complex form and, as such, are relegated to the appendix. In understanding these strategies, it is most useful to consider the utility offered to the median voter by the candidates' policy proposals and their respective valences (referred to in the Appendix as Platform-Valence Offers, or PVOs, and denoted by z). Indeed, the development of equilibrium in the Appendix proceeds entirely from this perspective, treating the policy proposals and valences as comprising the candidates' "bids" in a contest (with spillovers) for the voter's support.

Proposition 2. *Suppose the opportunity cost of contesting the election is neither too high nor too low relative to the probability that the potential challenger is a strong type (i.e., region 1 in Figure 2), such that both contested and uncontested elections may occur in equilibrium.*

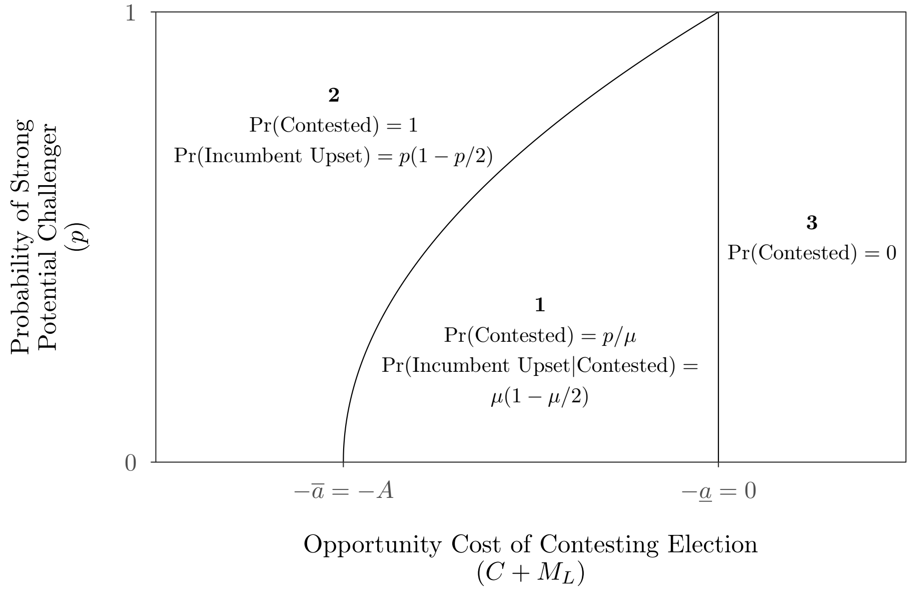
An increase in the probability that the potential challenger is a strong type (p) increases the probability that the election is contested and the probability that the incumbent is defeated.

An increase in the opportunity cost of contesting the election ($C + M_L$) lowers the probability that the incumbent is challenged and the probability that the incumbent is defeated but raises the probability that the incumbent is defeated conditional on having been challenged.

In the equilibrium case of greatest interest (case 1 in Figure 2), contested and uncontested elections both occur with strictly positive probability. While the strong type of potential challenger always contests the election, the weak type of potential challenger randomizes between entering the election and leaving the race uncontested. As such, the incumbent believes she faces a strong type with greater probability, denoted by μ, than the latent probability that the potential challenger is of strong type, p. Specifically, when challenged, the incumbent believes she faces a strong type with probability $\mu = \sqrt{(A + C + M_L)/A}$. The weak type of potential challenger's randomization with regard to the entry decision leaves the incumbent indifferent when challenged between proposing policies as though she faces a weak type and proposing policies as though facing a strong type.

If the weak challenger enters, he proposes $x_R = 0 = M_V$, a platform that offers the median voter a utility of 0. The incumbent (if challenged) and the strong challenger both randomize over an interval of policies. The most extreme policies (vis-à-vis the median voter's ideal point) in the support of their respective distributions of policy proposals are $x_L = -A$ and $x_R = A$, which offer the voter a utility of 0 given the candidates' valences.

Figure 2: Equilibrium Cases



Notes: The opportunity cost of contesting the election, consisting of the cost of entry C (not incurred unless the potential challenger enters) and the policy M_L (the incumbent's ideal point) at which the incumbent wins if the potential challenger does not contest the election, increases along the horizontal axis. The probability p that the potential challenger is of strong type is increasing along the vertical axis.
Within region 1, the equilibrium case of greatest interest, the challenger both wins upon entering and chooses not to contest the election with positive probability. In the event that the potential challenger contests the election, the incumbent's belief that her opponent is of strong type is given by $\mu = \sqrt{(A + C + M_L)/A}$. The unconditional probability that the incumbent is upset is given by $p(1 - \mu/2)$.

The incumbent proposes this policy with probability $1 - \mu$ when challenged, and she adopts a distribution of policy proposals that mirrors the strong challenger's with probability μ, the probability that she faces a strong type of challenger in equilibrium.[7] As such, the distributions of utility offered to the voter by the incumbent and the challenger distributions (the latter a weighted sum of the two types) are identical. The "degree of moderation" of these distributions of policy proposals will be said to have increased if, for each policy in the support, the probability of proposing a policy at least that far from the median voter's ideal point has weakly decreased.[8]

[7]The distributions of policy proposals leave the strong type of challenger and the incumbent indifferent among all of the proposals over which they randomize and leave the weak type of potential challenger indifferent between entering and not.

[8]Alternatively, the degree of moderation of a candidate's distribution of proposals may be said to have

Across all of the cases, the weak type of challenger never wins. A weak challenger can offer at most 0 in utility to the median voter, which is the minimum the incumbent will ever offer to the voter in equilibrium, and the voter breaks ties in favor of the incumbent over the weak challenger when they propose $x_L = -A$ and $x_R = 0$. Within case 1, the weak challenger is willing to enter only if the incumbent is sufficiently likely to propose policies as though she faces a strong type. Given the incumbent's distribution of policy proposals, the strong challenger wins half of the time that the incumbent mixes over proposals more moderate than $-A$ ($\mu/2$) and wins all of the time the incumbent plays as though she faces a weak type $(1-\mu)$. Conditional on the election being contested, with μ probability the challenger is of strong type, so there is a $\mu(1-\mu/2)$ chance that the incumbent loses. The probability that the election is contested is p/μ, which increases as a function of increased entry by weak potential challengers lowers μ.

Within case 1, if the probability nature chose a strong type were to increase from p to p', the incumbent would propose more moderate policies, as she believes she is more likely to face a stronger type. This would induce weak challengers to enter more frequently, however, which would mitigate the incumbent's incentive to converge. To maintain equilibrium, then, the weak challenger must enter somewhat more frequently, but all other strategies remain the same. The probability of a contested election rises, but there are no additional effects. An increase in p does not necessitate a change in μ in order to maintain equilibrium.

If the opportunity cost of contesting the election were to increase from $(C + M_L)$ to $(C + M_L)'$,[9] given the proposal strategy of the incumbent under $(C + M_L)$, weak challengers would strictly prefer to remain out of the race. In turn, this would lead to an increase in the moderation of the incumbent's distribution of proposals, who would be sure she faces a strong type, which would lead to entry by weak types again. Only through an increase in the moderation of the proposal distributions and a reduction in the frequency with which the weak challenger enters can equilibrium be maintained when the opportunity cost of contesting the election increases. The decrease in the likelihood of a contested election serves in equilibrium to increase the incumbent's belief that she faces a strong type (by way of reduced entry by weak types), and so more protective policy proposal strategies by the incumbent and strong types of challenger must accompany this change. Because it affects the incumbent's belief that she faces a strong type, an increase in the opportunity cost of contesting the election leads to fewer contested elections and fewer upsets, but a higher likelihood of an upset conditional on the election being contested.

Proposition 2 summarizes these mechanisms as predictions for the change in the frequency of contested

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increased if and only if the distribution of utilities offered to the voter (a function of the policy proposal and the candidate's valence) is increasing in the sense of first-order stochastic dominance.

[9]Via the cost of entry rising from C to C' or the incumbent's ideal point moving from M_L to a more moderate M_L'.

elections and incumbent defeats as a result of changes in the opportunity cost of entry or the strength of the pool of potential challengers. Perhaps more importantly, though, it establishes that the probability of the upsets conditional on the election being contested and the probability that the election is contested are not independent of one another. In fact, they are inversely related through the incumbent's equilibrium belief that she faces a strong type, which is a function of the opportunity cost of entry in equilibrium case 1.

The implication for empirical work is stark; the decision to restrict attention to only contested elections is not without consequence. Estimates of the effect of changes in either component of the opportunity cost of entry (the cost of entry or the winning policy in an uncontested race) on the likelihood that an incumbent retains her seat could be incorrectly signed. The bias would be so severe as to mischaracterize the most fundamental descriptor of a relationship. We explore the issues this poses for estimation in greater depth below when considering voter welfare.

Challenger Quality and Upset Victories

Animating the previous result is the role the incumbent's equilibrium belief that a challenger is strong, μ, plays in determining the frequency of various outcomes in equilibrium. While this is an endogenous quantity, itself a function of parameters A, C, and M_L, it most parsimoniously demonstrates the connections among contested elections, upset victories, and the degree of moderation in candidate proposals in contested elections. Consider the following decomposition of the probability that the incumbent, L, is upset by a potential challenger, R, who is a strong type S with probability p.

$$\begin{array}{rcccccccl} \Pr(R \text{ upsets } L) & = & \Pr(R \text{ wins}|t=S, \text{contested}) & \cdot & \Pr(t=S|\text{contested}) & \cdot & \Pr(R \text{ challenges } L) & ^{10} \\ & = & (1-\mu/2) & \cdot & \mu & \cdot & p/\mu & \end{array} \tag{5}$$

A contested election is of course a necessary condition for an upset to occur. When the election is contested, only strong challengers defeat incumbents in the model, and then only when the incumbent underestimates a strong challenger, believing him to be of weak type.

The next proposition highlights that the probability of a given strong challenger winning is inversely related to the incumbent's belief that a challenger is of strong type, while the probability of incumbent defeat depends positively on the equilibrium probability she faces a strong type. The result establishes that if the likelihood of an upset victory occurring in a contested election rises, the probability that a given strong challenger will win actually falls. The proposition applies to all cases in which entry occurs, which requires

[10] $\Pr(R \text{ wins}|t=W, \text{contested}) \cdot \Pr(t=W|\text{contested}) = 0 \cdot (1-\mu)$, so Equation 5 omits these terms.

only that the opportunity cost of entry not be too high (regions 1 and 2 in Figure 2).[11]

Proposition 3. *Suppose the opportunity cost of contesting the election is not too high, such that entry occurs.*

While an increase in the opportunity cost of contesting the election ($C + M_L$) or the likelihood that a potential challenger is of strong type (p) raises the probability that in a contested election the challenger defeats the incumbent increases, the probability that a given strong challenger wins decreases.

The greater the likelihood that the incumbent is facing a strong challenger, the less likely she will be to underestimate her opponent. As μ increases, then, a strong challenger is less likely to win. Although a higher equilibrium probability that a challenger is of strong type leads to a lower probability that a given strong challenger wins, the first effect outweighs the second. The probability of an upset occurring conditional on entry is still increasing in μ.

Empirical work on challenger quality employs observable measures, such as previous office-holding experience. While the model presented here is predicated on challenger strength being unobservable, the preceding result is nonetheless relevant to these studies. It is likely that observable signs of strength (e.g., experience) are positively (though imperfectly) correlated with unobservable qualities. A challenger with experience is likely drawn from a pool of challenger types with a higher p than the pool from which a challenger without experience was drawn. Either may be strong or weak types on unobserved factors, but it is more likely the experienced candidate is genuinely a strong type.

If unobservable aspects of strength are correlated with observable measures, then studies would find (as they have) that experienced challengers more frequently defeat incumbents. (As noted, a higher probability of nature choosing a strong type increases increase the frequency of upset victories in equilibrium.) Yet a strong challenger with previous experience holding office may be less likely to win than a strong challenger who has no previous office-holding experience to signal his unobservable quality. An individual strong challenger may, in fact, have worse prospects because the expected strength of the pool of experienced candidates causes the incumbent to be more cautious about diverging from the median voter's preferences. This suggests the need to allow for differential effects among groups of challengers based on the prior belief this group would field a strong candidate.

[11] Note that if the probability that a potential challenger is sufficiently high relative to the opportunity cost of contesting the election (region 2 in Figure 2), potential challengers of both types always enter, so $\mu = p$, and the probability of an upset conditional on entry is then equal to the unconditional probability of an upset: $p(1 - p/2)$.

Voter Welfare

The results above both suggest a nuanced relationship of voter welfare to the frequency of contested elections and upset victories. Although the probability of an upset victory rises at a muted rate relative to the share of challengers contesting the election that are strong types, the candidates' equilibrium proposal distributions are increasingly moderate. The next result delves more deeply into the implications of the model for voter welfare, taking into account the competing effects of changes in the equilibrium probability that the incumbent faces a strong type.[12]

Proposition 4. *Suppose the opportunity cost of contesting the election is neither too high nor too low relative to the probability that the potential challenger is a strong type (i.e., region 1 in Figure 2), so contested and uncontested elections may occur in equilibrium.*

Voter welfare is strictly increasing in p as long as the opportunity cost of contesting the election is not too high (i.e., regions 1 and 2).

Voter welfare in contested elections is increasing in C and M_L, however, the probability of an uncontested election is increasing in C and M_L, which exerts downward pressure on voter welfare.

The implication arising in this proposition is that voter welfare in contested elections is decreasing as the cost of entering the election falls.[13] This effect is only tempered by a corresponding increase in the likelihood that the election is contested. Conversely, raising the cost of entry to positively affect voter welfare in contested elections entails an increase in uncontested elections, which negatively impacts voter welfare. Which of these two effects dominates varies across the equilibrium case of interest in which uncontested elections and upsets occur with positive probability. Fundamentally, however, manipulating the cost of entry entails a trade-off in voter welfare with respect to the likelihood of an uncontested election and the expected moderation of the winning policy in contested elections.

A similar analysis applies when considering changes in the location of the incumbent's ideal point relative to the median voter's, with one key difference. Analogous to an increase in the cost of entry, a decrease in the distance between the incumbent's ideal point and the median voter's ideal point leads to more competitiveness in contested elections – a boon for voter welfare – but also more uncontested elections – the bane of voter welfare. As M_L grows closer to M_V, however, voter welfare in uncontested elections becomes less negative (closer to the utility the voter receives from electing an incumbent that proposed $x_L = -A$). Because of this additional implication of increasing M_L, the downward pressure on voter welfare of increasing the frequency

[12] Specifically, "voter welfare" denotes the utility of the median voter.

[13] Prato & Wolton (2017) uncover a similar theoretical result.

of uncontested elections is less than when increasing C. If an increase in C led to a net increase in voter welfare, then a corresponding increase in M_L would certainly lead to a net increase in voter welfare.

The most likely means by which the distance between the incumbent's and the median voter's ideal points would decrease is through an increase in the homogeneity of the constituency. Even though the frequency of uncontested elections increases, the incumbent's ideal point is more palatable to the median voter, and the distribution of policy proposals in contested elections is more moderate, so geographic sorting may contribute positively to the representation individuals receive. With fewer weak potential challengers contesting the election, the incumbent's belief that she faces a strong type when the election is contested increases, leading to more moderate distributions of policy proposals in contested elections.

Only an increase in p, the latent quality of the pool of potential challengers, always has the net effect of raising voter welfare. When the opportunity cost of contesting the election is large enough such that weak potential challengers randomize between entering the race and not (case 1), the probability that the election is contested increases without affecting the moderation of proposals in contested elections. When the opportunity cost of contesting the election is too small relative to the probability the potential challenger is of strong type – such that all potential challengers always enter (case 2) – the competitiveness of contested elections is increasing in p, and all elections are contested. The reduced probability of an upset is of secondary concern for voter welfare as it stems from the incumbent believing she more likely faces a strong challenger and, accordingly, proposing more moderate policies in expectation.

Earlier results suggested that excluding uncontested elections in studies of the frequency of challengers upsetting incumbents could suffer from severe bias, to the point of incorrectly estimating the sign of the coefficient on the cost of entry or the location of the incumbent's ideal point. Similarly, if an analysis excluded uncontested elections, then the effects on voter welfare of an increase in the cost of contesting an election or the incumbent's ideal point would be unambiguously positive, but this would ignore the negative effects on voter welfare of having more uncontested elections. The exclusion of uncontested races could significantly bias inference.

Suppose, for instance, that one wished to understand the effect on the distance of the winning policy from the median voter's ideal point as a function of the distance of the incumbent's ideal point from the median voter's. To simplify the thought experiment greatly, the analyst might run the following model:

$$|x - M_V| = \alpha + \beta \cdot |M_L - M_V| + \epsilon. \tag{6}$$

This analysis would be in the spirit of studies such as McCarty, Poole & Rosenthal (2006), who investigate the effect of gerrymandering on polarization, i.e., the effect of changes in the distribution of ideologies within a district on the extremism of the policies espoused by the representative of that district.

Were the sample used to test the model given in equation 6 to include only contested elections, the estimate of β would be negative. This would be a somewhat surprising conclusion, suggesting greater district-level polarization leads to less extreme policies. The model offers an explanation, namely, that a more homogenous district would lead to less entry. This in turn gives the incumbent a higher belief that she faces a strong type and thus leads to more moderate, i.e., protective policy proposals (and thus safer seats). Such an inference would be incomplete, however, as it excludes the relatively extreme winning policies that occur in uncontested elections, a crucial part of the mechanism by which greater moderation came about. If the ideological dispersion were sufficiently great in the constituency, the bias could even result in incorrectly signing the effect of interest.

Incorporating a Non-Policy Benefit to Holding Office

To this point, the model has supposed only policy-motivated candidates. In what ways would the analyses change if holding office also motivated the candidates to win the election? As the spoils of office increase, what is the effect on the frequency of uncontested elections, the likelihood of challenger upsets, and the degree of moderation of policy proposal distributions in elections that are contested?

Suppose candidates are office-motivated as well as policy-motivated, where winning the election confers a benefit of $B > 0$, representing the salary, professionalization, or prestige of holding office. As demonstrated by the more general version of the model derived in the Appendix, the structure of the equilibrium cases takes essentially the same form.[14] In the equilibrium case of interest, which features both uncontested elections and incumbent upsets, the benefit to holding office operates through the same mechanism as the opportunity cost of contesting the election $(C + M_L)$, although in the opposite direction.

An increase in B lowers the equilibrium probability μ that the incumbent faces a strong challenger in a contested election. At equilibrium, a higher non-policy benefit to holding office would provide an incentive for the incumbent to converge fully, which would lead to greater entry by weak types of potential challengers. Equilibrium requires greater entry by weak types, which lowers the incumbent's belief that she faces a strong type, leaving her indifferent among the interval of proposals over which she randomizes. The strong type of challenger, however, proposes policies according to a more moderate distribution of platforms. The

[14]Regions 1 and 2 change somewhat vis-à-vis their appearance in Figure 2 to accommodate subcases in which the incumbent converges entirely to the median with positive (even full) probability; this provides a greater incentive for the weak type of potential challenger to enter, and it reflects increased incumbent defensiveness over an increasingly valuable office.

incumbent's distribution of policy proposals is more moderate when she proposes policies as though she faces a strong type, but she more often proposes $x_L = -A$, as though she faces a weak type, so her entire distribution of proposals cannot be said to be more moderate.[15]

Because a decrease in B leads to an increase in μ, and as Proposition 5 in Appendix G makes clear, an increase in B leads to an increase in the probability of a contested election, increasing the overall probability of an upset but decreasing the probability of an upset conditional on the election being contested. As in Proposition 3, a given strong challenger fares better even as the likelihood of an upset in a contested election falls. Although the equilibrium probability increases that the incumbent faces a weak type increases and thus proposes policies as though she faces a weak type, the increased moderation of proposals from strong challengers and thus the incumbent when she proposes policies as though she faces a strong challenger leads to a net increase in voter welfare in contested elections. The expected winning policy in a contested election, however, is closer to the median voter's ideal point. Combined with the accompanying decrease in uncontested elections, an increase in B results in increased voter welfare.

The non-policy benefits of holding office have long been subject to scholarly attention. Squire (2000) examines uncontested elections at the level of state legislatures. He hypothesizes that increased professionalization within a legislature encourages entry by challengers, and he uncovers exactly this relationship. Further, the frequency of uncontested seats at the national level is even lower, where the prestige of holding office is even greater (Wrighton & Squire 1997). Within the model, and in line with these patterns, an increase in the benefit of holding office leads to greater entry. The increase in entry, however, is not a result of potential challengers finding the prospect of holding office more valuable. Instead, the increase in entry is driven by the weak challenger contesting the election with greater frequency. The weak type will never win, but in equilibrium the increase in entry serves to temper the incumbent's desire to converge enough to defend her seat and secure the benefit B with certainty.

A number of papers in recent years have used a regression discontinuity design in which small population shifts trigger discontinuous increases in the salary awarded to politicians to draw inference about electoral accountability and the incumbency advantage (Eggers, Freier, Grembi & Nannicini 2018). The theoretical quantity of interest, elected officials' salaries, corresponds the quantity B in our model. It bears noting again that whether or not such analyses include only contested elections could have dramatic effects on inference about the effect of B on outcomes such as incumbent retention and voter welfare.

Finally, consider the policy implications of the model, by way of the following thought experiment. Citizens may choose to reduce the cost of entry by publicly funding campaigns, or they may choose to increase the

[15]The distribution of utilities offered to the voter is not first-order stochastically increasing.

salary of elected officials. Which would they choose? The past few sections have demonstrated that each involves trade-offs in terms of uncontested elections and the moderation of policy among the elections that are contested, although increasing B performs somewhat better than decreasing C. Only an increase in the quality of the pool of challengers, p, unambiguously raises voter welfare, resulting in fewer uncontested elections and more moderate policy proposals by candidates in elections that are contested.

Conclusion

This paper presented a model of challenger entry followed by electoral competition, where the incumbent is initially uncertain as to the strength of a potential challenger. The model revealed that the incumbent will never know with certainty whether she faces a strong or a weak opponent until after she has laid down a policy stake, i.e., it is not the case that only strong or weak challengers enter in equilibrium. In the case of greatest interest, uncontested elections, upset victories, and varying levels of convergence in the winning policy all occur. The rich set of outcomes stem from what might be termed the incumbent's "rational complacency." The incumbent, valence-advantaged as she is, could moderate enough in policy to ensure victory. She is willing, though, to adopt less moderate positions, weighing the chance of winning at policies she finds more attractive against the possibility of losing to a strong challenger.

A key set of relationships emerges from the equilibrium strategies, namely, the connection between the probability that the election is contested and the behavior of candidates in contested elections. Specifically, exogenous changes that affect the equilibrium probability that the incumbent faces a strong type of challenger affect both the likelihood of contested elections as well as the frequency of upsets, which itself hinges on the extent to which candidates moderate in policy proposals towards the median voter. These parameters include the cost of contesting the election, the incumbent's ideal point (which is the winning policy in an uncontested election), and the non-policy benefit to holding office.

These countervailing effects become particularly salient when considering voter welfare. A decrease in the cost of entry or a more moderate incumbent ideal point lead the probability of a contested election to increase (a benefit to the median voter) while causing the expected policy proposals in contested elections to be farther from the median voter's ideal point (a loss for the median voter). A number of implications for empirical work followed, notably the potential bias induced by dropping uncontested elections from studies of incumbent retention or the polarization of winning policies. Comparative statics may change in magnitude or even in sign depending on the decision to consider all elections or only contested elections.

An increase in the prior probability that the challenger is strong is the only parameter change that does not entail trade-offs in voter welfare (although an increase in the non-policy benefit to holding office does result in a net increase in voter welfare). Contested elections increase, and proposals in contested elections become increasingly moderate. As such, recruiting stronger pools of potential candidates emerges as a recommendation of the model for those seeking to improve election outcomes by producing winning policies that are closer in expectation to the median voter in each constituency.

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Appendix

Results and Proofs Referenced In-Text

A Preliminary Results: Lemmas 1-3

In the context described in-text, the elimination of weakly dominated strategies implies that voters will vote sincerely, which in turn allows us to invoke the median voter theorem and restrict attention to the median voter's decision. The first result formalizes this, establishing that it is as though the incumbent and the challenger, L and R, compete only for the support of the median voter, V. The elimination of weakly dominated strategies will also help narrow the range of policy proposals we need to consider as we develop the players' equilibrium strategies below, which the second lemma makes explicit.

Lemma 1. *The majority preference relation is equivalent to the median voter's preference relation.*

Proof of Lemma 1. The assumption that players, including voters, do not play weakly dominated actions implies that If L is challenged by R, no voter votes for a candidate whose policy proposal, if implemented, and valence, if elected, would provide the voter strictly lower utility than the opposing candidate. Voting is sincere, with voters only voting for their most preferred candidate.

Voter preferences are supermodular in ideal points and the winning policy proposal, and this property is preserved with valence entering additively into voter utility. This is a sufficient condition to apply the results from Gans & Smart (1996), and the result follows immediately. ■

Lemma 2. *In any equilibrium, $x_L \in [M_L, 0]$ and $x_R \in [0, M_R]$.*

Proof of Lemma 2. We show that for any policy $x_L \notin [M_L, 0]$ or $x_R \notin [0, M_R]$, the candidates could weakly improve their chances of winning and/or the utility they would derive from winning by proposing a policy within those intervals.

Consider $x_L < M_L$. $\hat{x}_L = M_L$ increases the utility the incumbent offers the voter, thus weakly increasing L's chance of winning, and provides L more utility from the winning policy should she win. $\hat{\hat{x}}_L = 0$ similarly weakly dominates any proposal $x_L > 0$. A symmetric argument applies to R.

As such, the assumption to eliminate weakly dominated strategies narrows the domain of proposals available to L and R to the intervals given above. ■

A strategy for the incumbent is a policy to propose when unchallenged as well as a (possibly degenerate) distribution of policies according to which to propose when challenged, where in this latter case x_L is distributed according to the cumulative distribution function ξ_L. The support of this distribution will be a subset of the interval $[M_L, 0]$. A strategy for the challenger is a pair for each type consisting of a probability of entry, $\sigma_{R|t} = \Pr(enter|t)$, and a distribution of policy proposals, $x_{R|t}$, with distribution function $\xi_{R|t}$. The support of this distribution will be a subset of the interval $[0, M_R]$.[16] Let the incumbent's belief that she faces a strong type be denoted by μ, which by consistency of beliefs must equal $\frac{p\sigma_{R|S}}{p\sigma_{R|S}+(1-p)\sigma_{R|W}}$ when at least one of the types of challenger enters with strictly positive probability.

The possibility of voter indifference affects a candidate's expected utility from proposing x_c only to the extent that, given ξ_{-c}, there is a strictly positive probability that V will be indifferent the two candidates. However, two cases of voter indifference will be of particular importance going forward, and the next result concerns these cases. Specifically, in the event that a challenger of type t converges in policy entirely to the median voter's ideal point, 0, and the incumbent proposes $x_L = A_{R|t} - A_L$ (converging just enough to leave V indifferent), the next lemma says V must vote for L.

Lemma 3. *No equilibrium can exist in which a challenger of type t proposes $x_R = 0$ with strictly positive probability and the support of ξ_L includes $x_L = A_{R|t} - A_L$, but in which V does not vote for L in the event that the incumbent proposes $x_L = A_{R|t} - A_L$, a challenger of type t proposes $x_R = 0$, and $A_{R|t} - A_L \neq 0$ (i.e., if L has policies available to her at which she could win with certainty against a given type of challenger converging entirely to the median).*

Proof of Lemma 3. For $\xi_{R|t}$ such that $\Pr(x_R = 0|t) = \alpha > 0$, suppose that V votes for type t of R with probability $\beta > 0$ when $x_L = A_{R|t} - A_L < 0$ and type t of R has proposed $x_R = 0$.

Consider $\hat{x}_L = A_{R|S} - A_L + \epsilon$, where $\epsilon > 0$. Then $\mathbb{E}U_L(\hat{x}_L) > \mathbb{E}U_L(A_{R|S} - A_L)$ if $-M_L - C + A_L - A_{R|S} - \epsilon > -M_L - C + (1 - \mu\alpha\beta)(A_L - A_{R|S}) + \mu\alpha\beta \cdot (0) \Rightarrow \epsilon < \mu\alpha\beta(A_L - A_{R|S})$. Such an ϵ clearly exists, so it cannot be the case that $x_L = A_{R|S} - A_L$ was in the support of ξ_L in any equilibrium in which ties are not broken entirely in L's favor when $x_L = -\underline{a}$ and a strong challenger has proposed $x_R = 0$.

This same logic applies to $\xi_{R|W}$ s.t. $\Pr(x_R = 0|W) > 0$, except that we must account for the fact that it is possible that the strong type could be proposing $A_{R|S} - A_{R|W}$. This only makes the argument above stronger, however, as $\hat{\hat{x}}_L = A_{R|W} - A_L + \epsilon$ also reduces the likelihood of the strong challenger winning. ■

[16] We do not assume the distributions are continuously differentiable, but results below establish that the interiors of any non-degenerate equilibrium distributions of proposals lack mass points and gaps and so are, in fact, in C^1.

B No Entry Separation: Proposition 1

Proposition 1. *The two types of challengers never fully separate with respect to entry in equilibrium.*

Proof of Proposition 1. If the incumbent believes that only weak types of challengers enter in equilibrium, she would propose $x_L = A_{R|W} - A_L$, converging enough so that there is no policy available to the weak type such that voting for R offers the voter more utility than would voting for L would provide.[17] However, it would not be sequentially rational for the strong type to remain out of the election. Any $x_R < A_{R|S} - A_{R|W}$, which makes voting for S more attractive for V than voting for L, would win with certainty.

If the incumbent believes she faces only a strong type, then she would converge to $x_L = A_{R|S} - A_L$. As established V must break ties in favor of L for an equilibrium to exist, so R would be losing with certainty. If the strong type remains willing to contest the race, i.e., $C + M_L < A_{R|S} - A_L$, the weak type would also prefer to enter (getting more convergence from the incumbent than his low valence "deserves"). If the strong type is not willing to incur the cost of entry only to lose to an incumbent converged to $x_L = A_{R|S} - A_L$, then neither would the weak type. ∎

C Constructing Equilibria in Randomized Strategies: Lemmas 4-6

As discussed in the text, it is useful to consider the utility that a candidate offers to the voter with the candidate's policy proposal and associated valence when developing the policy proposal strategies in this model.[18] For instance, if L proposes x_L, the voter's utility from voting for L would be $x_L + A_L$. As such, let $z_L : [M_L, 0] \to \mathbb{R}$ simply be an affine transformation given by $z_L(x_L) = x_L + A_L$, the utility the voter would receive from electing L. Similarly, we characterize the utility offered to the median voter by the strong type with the function $z_{R|t} : [0, M_R] \to \mathbb{R}$ where $z_{R|t}(x_R) = -x_R + A_{R|t}$.

Since $x_L \sim \xi_L$, we write $z_L \sim \zeta_L$, where $\zeta_L(z) = \xi_L(z - A_L)$. Similarly, write $z_{R|t} \sim \zeta_{R|t}$, where $\zeta_{R|t}(z) = 1 - \xi_{R|t}(A_{R|t} - z)$. Denoting L's belief (correct on the equilibrium path) that the probability she faces a strong challenger is $\mu \in [0, 1]$, we may write $z_R \sim \zeta_R$ with $\zeta_R = \mu \cdot \zeta_{R|S} + (1 - \mu) \cdot \zeta_{R|W}$.

Definition We refer to z_c as comprising candidate c's *policy-valence offer* (PVO), and ζ_c as c's *distribution of PVOs*.

If c offers a PVO of $z_c > z_{-c}$, then the voter (voting sincerely by Lemma 1) will vote for candidate c. Note that conditional on winning, each candidate would prefer to do so at a PVO that offers the voter a lower

[17]Per Lemma 3, even if the weak type proposes $x_R = 0$, V must break the tie in favor of L in any equilibrium.

[18]In Hirsch & Shotts (2015), the authors refer to an analogous relationship as the score function.

level of utility, thus winning at a more divergent policy (i.e., a policy closer to the candidate's ideal point). Candidate c's utility conditional on winning is decreasing in z_c, though winning requires a sufficiently high z_c. As a first step in characterizing the equilibrium distributions of PVOs, the next lemma establishes that L and R's maximum PVO must be the same.

Lemma 4. *The support of both candidates' equilibrium distribution of PVOs must have the same maximum,* $\overline{z}$.

Recall that "candidate" here refers to L or R without drawing distinction between the possible types of challenger, i.e., both types' strategies need not include $\overline{z}$. Conditional on R winning, the type of challenger behind a given winning PVO certainly has implications for L's utility. For the sake of this result, however, it only matters to L whether or not some type of R is offering a higher PVO (i.e., more utility to the voter) than she is.

Proof of Lemma 4. Suppose that the candidates' maximum PVOs are such that $\overline{z}_c > \overline{z}_{-c}$. Then $\hat{z}_c = \frac{\overline{z}_c + \overline{z}_{-c}}{2}$ would yield c as high a probability of winning as z_c but at a policy closer to M_c. As such, it cannot be that $\overline{z}_c$ is part of c's distribution of PVOs in equilibrium, and so $\overline{z}_c = \overline{z}_{-c}$. ■

The challenger's highest possible PVO is $z_R = A_{R|S}$, if the challenger is of strong type and has converged in his policy proposal all the way to the voter's ideal point (i.e., R is of type S and proposes $x_R = M_V = 0$). As both candidates' distributions of PVOs must share the same maximum, $A_{R|S}$ is clearly an upper bound on $\overline{z}$.

Remark If L's equilibrium strategy entails a degenerate distribution of PVOs, her pure strategy must consist of offering $z_L = A_{R|S}$.

For any $\hat{z}_L < A_{R|S}$ offered with full mass, any candidate in whose favor V does not break ties at $\hat{z}_L$ would benefit from proposing a slightly more moderate policy and offering $z_c > \hat{z}_L$.

Remark Given Lemma 2, there must also exist finite minimum PVOs, $\underline{z}_c$. L's minimum offer cannot be less than $z_L = M_L - A_L$, while R's cannot be less than $z_R = -M_R - A_{R|S}$.

A related result is that if L is not converging to $x_L = A_{R|S} - A_L$ ($z_L = A_{R|S}$) with probability one, then the minimum of the support of both candidates' distributions of PVOs is equal to $\underline{z} = A_{R|W}$. Before proceeding to this lemma, consider that if L plays a mixed strategy, she must win with certainty at the maximum offer of utility, $\overline{z}$, that the candidates share. Either $\overline{z} = A_{R|S}$, and so L must win by Lemma 3, or $\overline{z} < A_{R|S}$ and $\Pr(z_R < \overline{z}) = 1$, by Lemma 4. Further, if L is randomizing in equilibrium, the strong type of R must win with at least some strictly positive probability at all policies (PVOs) over which she randomizes.

For the candidates to be willing to randomize over lower PVOs, they trade off a lower probability of winning and a smaller policy loss if they do win with winning more often at $\overline{z}$, a less attractive policy. Complicating this trade-off is that, the more divergent the policy position at which a candidate loses, the greater the expected disutility from the winning policy, i.e., the more divergent the opposing candidate's policy could have been while still offering the voter the greater level of utility, $z_{-c} > z_c$.

Lemma 5. *If both candidate's equilibrium strategies entail non-degenerate distributions of PVOs, z_c, then the minimum of the supports of these distributions, $\underline{z}_c$, must be the same, namely $\underline{z} = A_{R|W}$.*

Proof of Lemma 5. Any (type of) candidate whose equilibrium strategy entails a non-degenerate distribution of PVOs must either win with some strictly positive probability at the minimum of the distribution's support, or lose at all points in the support of the distribution. If $\overline{z} < A_{R|S}$, then both candidates win with certainty if they offer $z_c = A_{R|S}$. Even if $\overline{z} = A_{R|S}$ and L offers this PVO with positive mass, if she does not place full mass on it (i.e., $\Pr(z_L < \overline{z}) \in (0, 1)$), then R must clearly win with some probability if he offers $z_{R|S} = \overline{z} = A_{R|S}$, by the fact that the support of $\zeta_{R|S}$ must include $\overline{z}$ and because $\zeta_L(\overline{z}) > 0$.

As candidates randomize over less divergent policy proposals, they are trading off some probability of winning for the prospect of winning at a more appealing policy. In fact, they not only trade-off the probability of winning, though, but also incur greater expected disutility in expectation from the winning policy if they lose as they diverge. Nonetheless, for both of the candidates' equilibrium distributions to be non-degenerate, the candidates must be indifferent among all policies (or offers, z_c) over which they randomize, and so they must be winning with at least some strictly positive probability at the lowest PVO in their distributions. It must be, then, that $\underline{z}_L = \underline{z}_R$. If not, e.g., $\underline{z}_c < \underline{z}_{-c}$, then candidate c would lose with certainty at all $z \in [\underline{z}_c, \underline{z}_{-c})$, which we have argued cannot occur if ζ_c, ζ_{-c} are non-degenerate. ■

For candidate L and at least one type of R to be randomizing in equilibrium, it must be true that $\zeta_R(\underline{z}) = \mu\zeta_{R|S}(\underline{z}) + (1-\mu)\zeta_{R|W}(\underline{z}) > 0$ and $\zeta_L(\underline{z}) > 0$. This implies that $\Pr(z_L = \underline{z}) > 0$. Suppose L and at least one type t of R both place positive mass on $\underline{z}$ and both win with some strictly positive probability against the other. One or both of L and R of type t would find it worthwhile to offer slightly above $\underline{z}$ instead of ever offering $\underline{z}$. So it must be the case that whichever type of R is placing mass on $\underline{z}$ is losing with certainty, but this means that it cannot be the type of R that is randomizing. It must be that S randomizes, does not place positive mass on $\underline{z}$, and W loses with certainty, but the only $\underline{z}$ for which there could be no deviation is $\underline{z} = A_{R|W}$, where, by Lemma 3, we know V must vote for L if indifferent between L and type W of R. Finally, note that it must be true that V would vote for type S of R if the strong challenger proposes $x_R = A_{R|S} - A_{R|W}$ and the incumbent $x_L = A_{R|W} - A_L$, but the strong challenger will not offer $\underline{z}$ with positive probability, and so from L's perspective, she need only take into account the probability with which she will face a weak type of challenger, degenerately offering $\underline{z}$.

The final lemma asserts that the PVO distributions when the candidates are mixing are continuously differentiable over their interiors. The proof establishes that, if L is mixing, R must be as well, and that each will randomize over an interval of PVOs, $[A_{R|W}, \overline{z}]$. Furthermore, when mixing, neither candidate will place any positive mass on any offer except $A_{R|W}$ and, for L, the maximum, $\overline{z}$, but only if $\overline{z} = A_{R|S}$.

Lemma 6. *In any equilibrium, the incumbent does not put strictly positive mass on any $z_L \in (\underline{z}, A_{R|S})$ with strictly positive probability. If ζ_L is a non-degenerate distribution, $\zeta_{R|S}$ will not place strictly positive mass on any PVO $z_R \in (A_{R|W}, A_{R|S}]$.*

If the candidates' equilibrium distributions of PVOs, $\zeta_c, c = L, R$, are both non-degenerate, the distributions will also have no gaps, i.e., $\forall a, b \in [\underline{z}, \overline{z}]$ s.t. $b > a, \zeta_c(b) - \zeta_c(a) > 0$.

Proof. Note that if $\overline{z} < A_{R|S}$, which we know by the discussion above cannot occur unless L's strategy is non-degenerate, then neither candidate will offer the voter utility of $\overline{z}$ with strictly positive probability in equilibrium. Only if $\overline{z} = A_{R|S}$ may L place positive mass on $\overline{z}$ in an equilibrium involving non-degenerate distributions of PVOs.

Suppose by way of contradiction that candidate c has placed positive mass on $\hat{z} \in (\underline{z}, \overline{z})$ (i.e., either L placing strictly positive mass on some $\hat{z} \in (\underline{z}, \overline{z})$ or R placing strictly positive mass on some $\hat{z} \in (\underline{z}, \overline{z})$ if ζ_L is a non-degenerate distribution). By the definition of mass point, $\Pr(z_c \in (z^-, z^+)) > 0$, where $z^- < \hat{z} < z^+$, and so regardless of how V breaks a tie at $\hat{z}$, there exists a discontinuous increase in the probability of $-c$ winning by increasing her PVO from z^- to z^+. As such, $\exists \gamma$ s.t. $\mathbb{E}U_{-c}(z^-) < \mathbb{E}U_{-c}(z^+), \forall z^- \in (\hat{z}-\gamma, \hat{z}), z^+ \in (\hat{z}, \hat{z}+\gamma)$. It cannot be a best response if $\exists z_{-c} \in (\hat{z}-\gamma, \hat{z})$, implying that c should instead offer $\hat{z}-\alpha\gamma, \alpha \in (0, 1)$, so $\hat{z}$ cannot be part of c's best response. This proves the first part of the lemma.

To prove that there will not be gaps in the distributions of PVOs when they are both non-degenerate, suppose by way of contradiction that c's distribution lacks support over $(a, b) \in [\underline{z}, \overline{z}]$, but where a, b are in the support of ζ_c. Note that $b \leq \overline{z}$. Also note that if both candidates propose according to non-degenerate distributions, each trades winning more often at higher PVOs with losing more often at lower PVOs, which are more attractive to the candidate.

Because $z_{-c} = a$ offers $-c$ a strictly higher expected payoff than any $z_{-c} \in (a, b)$ (the same winning probability but at a PVO that is more attractive to $-c$), $\nexists z_{-c} \in (a, b)$. As we established that the offer distributions share the same minimum if both are non-degenerate, then the two distributions must share the same gaps, if any exist. However, $z_c = a$ offers a strictly higher payoff than $z_c = b$, so it cannot be that b is in the support of the distribution of either L or the strong type of R, a contradiction of the assumption that there exists a gap in the distributions. ∎

D Characterizing the Equilibrium Cases: Proposition 2

Proposition 2 derives an initial set of comparative statics about the equilibrium of the model, given the simplifying assumptions employed in the paper. The results follow from a more general set of assumptions, and we derive the equilibrium under those assumptions first, in Proposition 2'. This serves both to demonstrate the results do not arise idiosyncratically from the simplifying assumptions made and, since the extensions allowing office-motivated candidates also arises as a case of the more general version, most parsiminiously derive the cases necessary for later results.

Before proceeding to fully characterize the equilibrium cases, we recall and establish some notation: the opportunity cost of contesting the election is given by $C + M_L =: \kappa$, $\underline{a} := A_L - A_{R|S} \geq 0$, $A_{R|S} > A_{R|W}$, $\overline{a} := A_L - A_{R|W} > 0$, $\sigma_{R|t}$ denotes the probability that a challenger of type t enters the election, and μ denotes L's belief that she faces a strong challenger.[19]

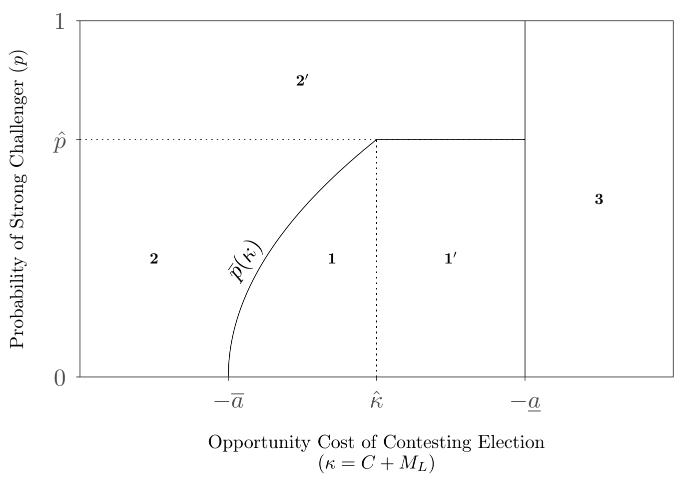
We continue to employ the "score function" given by z, which denotes the utility offered to the median voter by a candidate's valence and binding policy proposal. $\overline{z}$ is the maximum of the distribution of platform-valence offers (PVO) made by the candidates in equilibrium. Several additional functions appear in the statement of Proposition 2'. Their definitions and rules appear in the proof.

[19] From Lemma 3, we know that $\sigma_{R|W} > 0 \Rightarrow \sigma_{R|S} = 1$, so we know $\mu = \frac{p}{p+(1-p)\sigma_{R|W}}$ when the election is contested.

Proposition 2'. *The table, figure, and notes below characterize the strategies, by case, constituting an equilibrium of the model:*

Case	$\sigma^*_{R\|W}$	$\sigma^*_{R\|S}$	$z^*_{R\|W} \sim$	$z^*_{R\|S} \sim$	$z^*_L \sim$	$\bar{z}$	μ^*
1	$\frac{p(1-\mu^*)}{(1-p)\mu^*}$	1	$[A_{R\|W}]$	$\zeta^*(z;\bar{z})$	$(1-\mu^*)[A_{R\|W}] \oplus \mu^*\zeta^*(z;\bar{z})$	$Z(\kappa)$	$\bar{p}(\kappa)$
$1'$	$\frac{p(1-\mu^*)}{(1-p)\mu^*}$	1	$[A_{R\|W}]$	$\zeta^*(z;\bar{z})$	$\frac{-\underline{a}-\kappa}{-\underline{a}-\hat{\kappa}}\left[(1-\mu^*)[A_{R\|W}] \oplus \mu^*\zeta^*(z;\bar{z})\right] \oplus \frac{\kappa-\hat{\kappa}}{-\underline{a}-\hat{\kappa}}[A_{R\|S}]$	$A_{R\|S}$	$P(A_{R\|S})$
2	1	1	$[A_{R\|W}]$	$\zeta^*(z;\bar{z})$	$(1-\mu^*)[A_{R\|W}] \oplus \mu^*\zeta^*(z;\bar{z})$	$P^{-1}(p)$	p
$2'$	1	1	$[A_{R\|W}]$	$\frac{\hat{p}(1-p)}{(1-\hat{p})p}\zeta^*(z;\bar{z}) \oplus \frac{p-\hat{p}}{(1-\hat{p})p}[A_{R\|S}]$	$[A_{R\|S}]$	$A_{R\|S}$	p
3	0	0	$[A_{R\|W}]$	$[A_{R\|S}]$	$[A_{R\|S}]$	$A_{R\|S}$	1

The $z^*_L \sim$ column provides the distribution of L's proposals if R contests the race. Otherwise $z^*_L = M_L - A_L$ (i.e., $x^*_L = M_L$), and V votes for L. If $z_L = z_{R|t} = A_{R|t}$, V votes for L; given $z_L = z_{R|S} = A_{R|W}$, V votes for R; in other cases of indifference, let V vote for L with probability $\frac{1}{2}$.



$\hat{\kappa} := B - \sqrt{(A_L - A_{R|S} + B)(A_L + A_{R|S} - 2A_{R|W} + B)}$,

$\bar{p}(\kappa) = \frac{2(A_L - A_{R|W} + \kappa)}{A_L + A_{R|S} - 2A_{R|W} + B}$, $\hat{p} := 1 - \frac{\sqrt{A_L - A_{R|S} + B}}{\sqrt{A_L + A_{R|S} - 2A_{R|W} + B}}$

Notes on Notation:

- Let $q[Y] \oplus (1-q)\mathcal{Y}(\cdot)$ denote a strategy distributed according to the mixture of the actions given by Y (degenerate) and $\mathcal{Y}(\cdot)$ (non-degenerate distribution function), weighted by q and $1-q$, respectively.
- $\zeta^*(z;\bar{z}) = \frac{(1-P(\bar{z}))\left(\sqrt{A_L + A_{R|S} - 2A_{R|W} + B} - \sqrt{A_L + A_{R|S} - 2z + B}\right)}{P(\bar{z})\sqrt{A_L + A_{R|S} - 2z + B}}$, $\forall z \in [A_{R|W}, \bar{z}]$.
- $Z(\kappa) = A_{R|W} +$ $\sqrt{A_L - A_{R|W} + \kappa}\left(\sqrt{2(A_L + A_{R|S} - 2A_{R|W} + B)} - \sqrt{A_L - A_{R|W} + B}\right)$
- $P(Z) = 1 - \frac{A_L + A_{R|S} - 2Z + B}{A_L + A_{R|S} - 2A_{R|W} + B}$
- $\bar{p}(\kappa) = P(Z(\kappa))$
- The highest utility that the challenger could offer the voter is $A_{R|S}$, providing an upper bound for $\bar{z}$ and defining $\hat{p} = P(A_{R|S})$ and $\hat{\kappa} = Z^{-1}(A_{R|S})$.

Figure 3: Equilibrium Cases in the More General Version of the Model

Proof of Proposition 2'. Given Lemma 5, several cases may obtain, numbered according to the labels applied in Figure 3: 1 & 1') L and R could both propose policies according to non-degenerate distributions and the weak type of challenger could leave the election uncontested with some strictly positive probability, 2) L and R could both propose policies according to non-degenerate distributions and the weak type of challenger could always enter, 2') L could propose a degenerate distribution in which $x_L = A_{R|S} - A_L$ with R possibly still proposing policies according to a non-degenerate distribution, or 3) neither type of R could enter the election.

To preclude entry by both types, costs must be sufficiently high, $\kappa \geq -\underline{a}$, such that even if L converges as though she is certain she faces the strong type, a challenger would not find it worthwhile to incur the cost of entry. The beliefs which support equilibrium case 3 are $\mu = 1$, such that L is certain she faces a strong type of challenger off the equilibrium path and would respond to entry with a policy convergent enough to ensure victory against a challenger of strong type.

Lemma 6 guarantees that $\zeta_c, c = L, R$ will be either degenerate or continuously differentiable with pdf ζ_c'. If the latter is true, we may write the probability that any candidate wins at a platform-valence offer (PVO) $z_c \in (A_{R|W}, \overline{z})$ as $\zeta_{-c}(z_c)$. Recall that $\zeta_c(\overline{z}) = 1$, and if $\overline{z} < A_{R|S}$, then $\Pr(z_c = \overline{z}) = 0$.

We now consider the cases in which challenger entry occurs, and specifically the possibility that the two candidates' distributions of PVOs are both non-degenerate (cases 1 and 2). Arguments above establish it could either be the case that $\overline{z} < A_{R|S}$, where neither puts positive mass on $\overline{z}$, or that $\overline{z} = A_{R|S}$, with L offering $\overline{z}$ with potentially positive (but not full) probability but still randomizing (along with the challenger). We first suppose that $\overline{z} < A_{R|S}$. Indifference conditions are as follows, where $V(\cdot)$ denotes utility in terms of PVOs, z, such that $V_{c|t}(z_{c|t}(x_c)) = U_c(x_c, x_{-c}; t, -t)$.

$$
\begin{aligned}
\mathbb{E}V_L(z_L) &= M_L + [1 - \mu + \mu\zeta_{R|S}(z_L)] \cdot [A_L - z_L + B] + \mu \int_{z_L}^{\overline{z}} (z_{R|S} - A_{R|S})\zeta_{R|S}'(z_R)dz_{R|S} - C \\
&= M_L + A_L - \overline{z} + B - C, \;\; \forall z_L \in [A_{R|W}, \overline{z}] \\
\mathbb{E}V_{R|S}(z_{R|S}) &= -M_R + \zeta_L(z_{R|S}) \cdot [A_{R|S} - z_{R|S} + B] + \int_{z_{R|S}}^{\overline{z}} (z_L - A_L)\zeta_L'(z_L)dz_L - C \\
&= -M_R + A_{R|S} - \overline{z} + B - C, \;\; \forall z_{R|S} \in [A_{R|W}, \overline{z}] \\
\mathbb{E}V_{R|W}(A_{R|W}) &= -M_R + \zeta_L(A_{R|W})[A_{R|W} - A_L] + \int_{A_{R|W}}^{\overline{z}} (z_L - A_L)\zeta_L'(z_L)dz_L - C \\
&= -M_R + M_L
\end{aligned}
$$

We have three degrees of freedom, in a manner of speaking, namely: μ, $\overline{z}$, and $\zeta_L(A_{R|W}) \geq 0$. These may be adjusted to satisfy the three equalities, where it must also be the case that ζ_{-c} must leave c's expected utility constant for all $z_c \in [A_{R|W}, \overline{z}]$. Examining the second equality (evaluated at $z_{R|S} = A_{R|W}$) and the third equality, we see it must be the case that $\kappa - \zeta_L(A_{R|W})[A_{R|W} - A_L] = A_{R|S} - \overline{z} + B - \zeta_L(A_{R|W})[A_{R|S} - A_{R|W} + B] \Rightarrow \zeta_L(A_{R|W}) = \frac{(A_{R|S} - \overline{z} + B) - \kappa}{A_L + A_{R|S} - 2A_{R|W} + B}$. Mathematica's `DSolve[]` solves the differential equation that pins down the distribution $\zeta(z; \overline{z})$ that appears in the statement of the Proposition. The initial conditions of $\zeta_L(\overline{z}) = 1 = \zeta_{R|S}(\overline{z})$ and $\zeta_{R|S}(A_{R|W}) = 0$ yield relationships between μ and $\overline{z}$ as well as $\overline{z}$ and κ.

For the latter, consider a function of the opportunity cost of contesting the election that specifies what the maximum of the distribution of PVOs ($\overline{z}$) must be in order to support the indifference conditions necessary for a mixed-strategy equilibrium, $Z : [-\overline{a}, \overline{\kappa}] \longrightarrow [A_{R|W}, A_{R|S}]$. Specifically,

$$Z(\kappa) = -A_L + 2A_{R|W} - \kappa + \sqrt{2(A_L + A_{R|S} - 2A_{R|W} + B)(A_L - A_{R|W} + \kappa)}.$$

This is a strictly increasing function in κ, so we may define $\hat{\kappa} := Z^{-1}(A_{R|S})$ to be the value of κ at which $Z(\kappa)$ is equal to $A_{R|S}$, the upper bound of $\overline{z}$.

Further, let $P : [A_{R|W}, A_{R|S}] \longrightarrow [0, 1]$ give the highest proportion of strong potential challengers in the population for which, given $\overline{z} \in [A_{R|W}, A_{R|S}]$, there exists some level of entry by weak types, $\sigma_{R|W} \in [0, 1]$, that could support beliefs held by L that would sustain a mixed-strategy equilibrium. Entry by weak challengers decreases μ so that L remains willing to randomize. P is a function of $\overline{z}$, and is given by:

$$P(\overline{z}) = 1 - \frac{\sqrt{A_L + A_{R|S} - 2\overline{z} + B}}{\sqrt{A_L + A_{R|S} - 2A_{R|W} + B}}.$$

Note that P is a strictly increasing function in $\overline{z}$, and set $\hat{p} := P(A_{R|S})$. Let $\overline{p}(\kappa) := P(Z(\kappa))$, represented by the curve in Figure 3 separating cases 1 and 2, with $\overline{p} : [-\overline{a}, \hat{\kappa}] \longrightarrow [0, \hat{p}]$.

In case 1, the weak type of potential challenger strictly randomizes between entering and not such that $\sigma_{R|W} = \frac{p(1-\mu^*)}{(1-p)\mu^*}$ with $\mu^* = \overline{p}(\kappa)$. In case 2, the weak type of potential challenger always enters, implying $\mu^* = p$. The strong type of challenger and the incumbent randomize as per the distributions specified above. The presence of an upper bound on $\overline{z}$, which yield the threshold values $\hat{\kappa}$ and $\hat{p}$, suggest a few natural limits on cases 1 and 2. These cases are more similar than not, so they are denoted by 1' and 2'.

In 1', $p \in (0, \hat{p}]$ and $\kappa > B - \sqrt{(A_L - A_{R|S} + B)(A_L + A_{R|S} - 2A_{R|W} + B)} =: \overline{\kappa} =: Z^{-1}(A_{R|S}) \Rightarrow \overline{z} > A_{R|S}$.[20] In effect, indifference requires the strong challenger and incumbent to offer the median voter a utility higher than is possible for the strong challenger converging to the median's ideal point. The upper bound on the PVOs is too low to keep S indifferent among $z_{R|S} \in [A_{R|W}, A_{R|S}]$ while also keeping W indifference between staying out of the election and entering at $x_R = 0$ ($z_{R|W} = A_{R|W}$). In this case, the incumbent must

[20]Note that $\overline{\kappa} \in (-\overline{a}, -\underline{a})$, so κ s.t. $\overline{z} > A_{R|S}$, does not imply that neither type of challenger would be willing to enter even against L converging to $x_L = -\underline{a}$, as though only facing strong types.

place extra mass on convergence to $x_L = A_{R|S} - A_L = -\underline{a}$ ($z_L = A_{R|S}$), the most convergent proposal in the support of her equilibrium distribution of proposals, in order to maintain the indifference that supports the challenger strategies employed in case 1.

In 2', the incumbent converges to $x_L = -\underline{a}$ ($z_L = A_{R|S}$) with full probability, and the strong challenger mixes between the distribution of proposals that characterized his play in case 2 and $x_R = 0$ ($z_{R|S} = A_{R|S}$). We must verify that this leaves L indifferent among any $z_L \in [A_{R|W}, A_{R|S})$, but with a strictly higher payoff from $z_L = A_{R|S}$. From the indifference condtions when $p = \hat{p}$, we have that $\mathbb{E}(z_{R|S} - A_{R|S}; \zeta_{R|S}) = \frac{A_L - A_{R|S} + B - (1-\hat{p})(A_L - A_{R|S} + B)}{\hat{p}}$. L's expected utility at $z_L = A_{R|W}$ is equal to her expected utility at all $z_L \in (A_{R|W}, A_{R|S})$ by the construction of S's strategy. So $\mathbb{E}V_L(A_{R|W}) = (1-p)(A_L - A_{R|W} + B) + p\frac{A_L - A_{R|S} + B - (1-\hat{p})(A_L - A_{R|S} + B)}{\hat{p}} + \frac{p - \hat{p}}{p(1-\hat{p})} \cdot (A_{R|S} - A_{R|S}), \forall z \in [A_{R|W}, A_{R|S})$ given p. It is easy to verify this yields lower utility for L than $z_L = A_L - A_{R|S} + B$ if $\hat{p} < p$, which defines the present equilibrium case. Having derived strategies for case 2', this completes the proof.[21] ■

Proposition 2. *Suppose the opportunity cost of contesting the election is neither too high nor too low relative to the probability that the potential challenger is a strong type (i.e., region 1 in Figure 2), such that both contested and uncontested elections may occur in equilibrium.*

An increase in the probability that the potential challenger is a strong type (p) increases the probability that the election is contested and the probability that the incumbent is defeated.

An increase in the opportunity cost of contesting the election ($C + M_L$) lowers the probability that the incumbent is challenged and the probability that the incumbent is defeated, but raises the probability that the incumbent is defeated conditional on having been challenged.

Proof of Proposition 2. Setting $A_L - A_{R|S} = A_{R|W} = 0$ and $B = 0$ in Proposition 2', we have

$$\mu = \sqrt{(A + C + M_L)/A} \Rightarrow \frac{\partial \mu}{\partial C} = \frac{\partial \mu}{\partial M_L} = \frac{1}{2\sqrt{(A + C + M_L)A}} > 0,$$

i.e., μ is increasing in both components of the opportunity cost of contesting the election, C and M_L.[22]

Pr(R contested election) $= p/\mu$ is increasing in p and decreasing in μ, so it is decreasing in C and M_L.

Pr(R upsets L) $= p(1 - \mu/2)$ is increasing in p and decreasing in μ, so it is decreasing in C and M_L.

Pr(L upset by $R|R$ contests the election) $= \mu(1 - \mu/2)$ does not depend on p but is increasing in μ, so it

[21] We may back out the distributions of policy proposals from the distributions of PVOs given $\xi_L(x_L) = \zeta_L(x_L + A_L)$, and $\xi_{R|S}(x_R) = 1 - \zeta_{R|S}(A_{R|S} - x_R)$, although the distributions of PVOs are in fact all that is necessary to ascertain the frequency of various outcomes and determine voter welfare in equilibrium.

[22] Note that under the simplifying assumptions, $\hat{p} = 1$ and $\hat{\kappa} = -\underline{a}$.

is increasing in C and M_L. ■

E Challenger Quality and Incumbent Upsets: Proposition 3

Proposition 3. *Suppose the opportunity cost of contesting the election is not too high such that entry occurs with positive probability.*

While an increase in the opportunity cost of contesting the election ($C + M_L$) or the likelihood that a potential challenger is of strong type (p) raises the probability that in a contested election the challenger defeats the incumbent increases, the probability that a given strong challenger wins decreases.

Proof of Proposition 3. Again setting n cases 1 and 2, $\Pr(\text{upset}|\text{contested}) = \mu(1 - \mu/2)$ and $\Pr(\text{upset}|t = S) = \mu/2 + (1 - \mu) = (1 - \mu/2)$. The former is increasing in μ, the latter is decreasing in μ, where μ is increasing in $C + M_L$ in case 1 and increasing in (in fact, equal to) p in case 2. ■

F Voter Welfare: Proposition 4

Proposition 4. *Suppose the opportunity cost of contesting the election is neither too high nor too low relative to the probability that the potential challenger is a strong type (i.e., region 1 in Figure 2), so contested and uncontested elections may occur in equilibrium.*

Expected voter welfare in contested elections is increasing in C and M_L, however, the probability of an uncontested election is increasing in C and M_L, which exerts downward pressure on voter welfare.

Expected voter welfare is strictly increasing in p as long as the opportunity cost of contesting the election is not too high (i.e., regions 1 and 2).

Proof of Proposition 4. Per Proposition 2', in cases 1 and 2, conditional on a contested election, the degree of moderation is increasing in μ, as is the probability that L will play as though she faces a strong type. As such,

$$\begin{aligned}
\mathbb{E}(U_V|\text{contested election}) &= (1-\mu)^2 \cdot 0 + 2\mu(1-\mu)\int_0^{\overline{z}} z\zeta'_{R|S}(z)dz + \mu^2 \int_0^{\overline{z}} 2z\zeta'_{R|S}(z)\zeta_{R|S}dz \\
&= A\left((2-\mu)\mu + (1-\mu)^2 \ln\left[(1-\mu)^2\right]\right)
\end{aligned}$$

is increasing in μ.[23] In case 1, μ is increasing in C and M_L, while in case 2, $\mu = p$, so expected voter welfare in contested elections is increasing in C, M_L, and p.

[23]Mathematica code for verification to be made available with the appendix.

In case 2, there are no uncontested elections, so overall expected voter welfare is as given above, and increasing in p. In case 1, overall expected voter welfare is given by

$$\mathbb{E}(U_V) = \left(1 - \frac{p}{\mu}\right) \underbrace{(M_L + A)}_{-} + \frac{p}{\mu} \underbrace{\mathbb{E}(U_V|\text{contested election})}_{+},$$

which is increasing in p. An increase in C or M_L, however, puts more weight (through an associated increase in μ) on a strictly negative component of expected voter welfare and less weight on the strictly positive component. This trade-off always exists, even though $\mathbb{E}(U_V|\text{contested election})$ grows more positive in μ and even though $M_L + A$ is increasing in M_L, because of the assumption that $M_L < -A$; an increase in M_L or C still takes weight away from a positive term and places is it on a negative term. ■

G Office- and Policy-Motivated Candidates: Proposition 5

Proposition 5. *Suppose the opportunity cost of contesting the election is neither too high nor too low relative to the probability that the potential challenger is a strong type (i.e., region 1 in Figure 3), such that both contested and uncontested elections may occur in equilibrium.*

An increase in the benefit to holding office B raises the probability that the incumbent is challenged and the probability that the incumbent is defeated, but lowers the probability that the incumbent is defeated conditional on having been challenged. The probability a given strong challenger wins, however, is increasing in B.

Voter welfare in contested elections is increasing in B, as is overall voter welfare.

Proof of Proposition 5. Setting $\overline{a} = A$, $\underline{a} = 0$, and $B > 0$, we focus on case 1 of Proposition 2'.

As μ is decreasing in B, the results from Propositions 2-3 apply with the implications opposite those of C and M_L.

When deriving the analogous results of Proposition 4, we have

$$\mathbb{E}(U_V|\text{contested election}) = \frac{1}{2}(2A + B)\left((2-\mu)\mu + (1-\mu)^2 \ln\left[(1-\mu)^2\right]\right),$$

where the coefficient is increasing in B while the remainder is decreasing in B via μ. The expected voter welfare conditional on the election being contested, though, is net increasing in B. Further, as an increase in B puts more weight on the contested election outcome and less on the uncontested election outcome through μ, overall voter welfare is increasing in B. ■